THE STATE OF STRESS IN A THICK PLATE

(NAPRIAZHENNOE SOSTOIANIE TOLSTOI PLITY)
PMM Vol. 31, No. 2, 1967, pp. 230-241
I. I. VOROVICH and O. S. MALKINA
(Rostov-on-Don)
(Received October 29, 1966)

In this paper the solution is given for the problem of equilibrium of a thick plate when the plane faces are unloaded and tractions are specified on the cylindrical part of the boundary, the tractions being symmetric with respect to the middle plane. The behavior of the solution is studied for small values of the thickness parameter of the plate λ. It is shown that the state of stress within the plate is described by a certain biharmonic function for which the boundary conditions coincide with the Kolosov-Muskhelishvili conditions only to the first approximation. On the boundary the additional state of stress contains terms of the same order in λ as the solution of the two-dimensional problem That is, the two-dimensional problem does not give the true state of stress on the boundary even for a plate of very small thickness.

1. We shall examine a plate of thickness $2h$ made of an isotropic, homogeneous material. The boundary of the body is formed by two plane regions Γ_1 and the cylindrical surface Γ_2 (Fig. 1). The plane faces are considered to be free of load. This assumption is not vital, inasmuch as these tractions can always be removed by solution of the corresponding problem for an infinite layer (see, e. g., Lur'e [1 and 2]. We shall study the case in which the cylindrical part of the plate boundary is loaded by a self equilibrating system of tractions which are symmetrical with respect to the middle surface. This, along with the results of Aksentian and Vorovich [3], allows us to obtain the strain distribution in a plate in the general case.

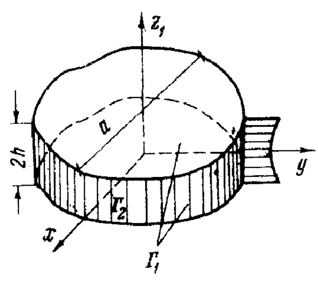

Fig. 1

We shall proceed from the three-dimensional Navier equations. It can be proved that in this case the state of stress in the plate is composed of three states, which are called states of biharmonic, potential, and curl type [2]. Using the results of [1], we obtain the state of stress of biharmonic type in the form

$$2\mu\,(u^{(1)}+iv^{(1)}) = \varkappa\varphi - z\overline{\varphi'} - \overline{\psi} - 2\,\frac{\nu-1}{3\nu-1}\,\lambda^2\zeta^2\left(\frac{1}{3}-\zeta^2\right)\overline{\varphi''}$$

$$w^{(1)} = -\frac{1}{\mu}\,\frac{\nu-1}{3\nu-1}\,\lambda a\zeta\,(\varphi'+\overline{\varphi'}) \qquad (1.1)$$

$$\sigma_x^{(1)} = \frac{1}{2}\,(2\varphi' + 2\overline{\varphi'} - \overline{z}\varphi'' - z\overline{\varphi''} - \psi' - \overline{\psi'}) - \frac{\nu-1}{3\nu-1}\,\lambda^2 a^2\left(\frac{1}{3}-\zeta^2\right)(\varphi''' + \overline{\varphi'''})$$

$$\sigma_y^{(1)} = \frac{1}{2}\,(2\varphi' + 2\overline{\varphi'} + \overline{z}\varphi'' + z\overline{\varphi''} + \psi' + \overline{\psi'}) + \frac{\nu-1}{3\nu-1}\,\lambda^2 a^2\left(\frac{1}{3}-\zeta^2\right)(\varphi''' + \overline{\varphi'''})$$

$$\tau_{xy}^{(1)} = -\frac{1}{2}\,i\,(\overline{z}\varphi'' - z\overline{\varphi''} + \psi' - \overline{\psi'}) - \frac{\nu-1}{3\nu-1}\,\lambda^2 a^2\left(\frac{1}{3}-\zeta^2\right) i\,(\varphi''' - \overline{\varphi'''})$$

$$\tau_{xz}^{(1)} = \tau_{yz}^{(1)} = \sigma_z^{(1)} = 0 \qquad (1.2)$$

$$\lambda = \frac{h}{a}, \qquad \zeta = \frac{z_1}{h}, \qquad \varkappa = \frac{5\nu + 1}{3\nu - 1}, \qquad \nu = \frac{1}{1 - 2\sigma}$$

where a is a characteristic linear plan dimension of the plate, ζ is a dimensionless coordinate, σ is Poisson's ratio, and $\varphi(z)$ and $\Psi(z)$ are analytic functions of the complex variable $z = x + iy$.

The state of stress of potential type can be found comparatively simply by seeking it in the form

$$u^{(2)} = \alpha(\zeta) \frac{\partial A}{\partial x}, \qquad v^{(2)} = \alpha(\zeta) \frac{\partial A}{\partial y}, \qquad w^{(2)} = \delta\beta(\zeta) A$$

After substitution of these expressions into the system of Navier equations, we may separate variables if A satisfies the relation

$$\partial^2 A / \partial x^2 + \partial^2 A / \partial y^2 - \delta^2 A / h^2 = 0 \qquad (1.3)$$

From the same procedure we find the function $\alpha(\zeta)$ and $\beta(\zeta)$ up to multiplicative constants. From the condition of homogeneity, i. e., from the fact that the plane faces are unloaded, we obtain the equation for the determination of the constant δ

$$2\delta + \sin 2\delta = 0 \qquad (1.4)$$

This has a countable set of complex roots, and to each root δ_k there corresponds a function A_k from (1.3).

Introducing the dimensionless coordinates ξ and η, we can represent the solution of potential type in the form

$$u^{(2)} = \lambda \sum_{k=1}^{\infty} \alpha_k(\zeta) \frac{\partial A_k}{\partial \xi}, \qquad v^{(2)} = \lambda \sum_{k=1}^{\infty} \alpha_k(\zeta) \frac{\partial A_k}{\partial \eta}, \qquad w^{(2)} = \sum_{k=1}^{\infty} \beta_k(\zeta) A_k$$

$$\sigma_x^{(2)} = \frac{2\mu}{a} \frac{\nu - 1}{\lambda} \sum_{k=1}^{\infty} p_k(\zeta) A_k + \frac{2\mu}{a} \lambda \sum_{k=1}^{\infty} \alpha_k(\zeta) \frac{\partial^2 A_k}{\partial \xi^2} \qquad \left(\xi = \frac{x}{a}\right)$$

$$\sigma_y^{(2)} = \frac{2\mu}{a} \frac{\nu - 1}{\lambda} \sum_{k=1}^{\infty} p_k(\zeta) A_k + \frac{2\mu}{a} \lambda \sum_{k=1}^{\infty} \alpha_k(\zeta) \frac{\partial^2 A_k}{\partial \eta^2} \qquad \left(\eta = \frac{y}{a}\right)$$

$$\tau_{xy}^{(2)} = \frac{2\mu}{a} \lambda \sum_{k=1}^{\infty} \alpha_k(\zeta) \frac{\partial^2 A_k}{\partial \xi \partial \eta}, \qquad \tau_{xz}^{(2)} = \frac{2\mu}{a} \nu \sum_{k=1}^{\infty} \gamma_k(\zeta) \frac{\partial A_k}{\partial \xi}$$

$$\tau_{yz}^{(2)} = \frac{2\mu}{a} \nu \sum_{k=1}^{\infty} \gamma_k(\zeta) \frac{\partial A_k}{\partial \eta}, \qquad \sigma_z^{(2)} = \frac{2\mu}{a} \frac{\nu}{\lambda} \sum_{k=1}^{\infty} s_k(\zeta) A_k \qquad (1.5)$$

where

$$\alpha_k(\zeta) = \left(\frac{\sin \delta_k}{\delta_k} - \nu \cos \delta_k\right) \cos \delta_k \zeta - \nu \zeta \sin \delta_k \sin \delta_k \zeta \qquad (1.6)$$

$$\beta_k(\zeta) = [(1 + \nu) \sin \delta_k + \nu \delta_k \cos \delta_k] \sin \delta_k \zeta - \nu \delta_k \zeta \sin \delta_k \cos \delta_k \zeta$$

$$p_k(\zeta) = \delta_k \sin\delta_k \cos\delta_k\zeta$$

$$\gamma_k(\zeta) = \delta_k(\cos\delta_k \sin\delta_k\zeta - \zeta\sin\delta_k\cos\delta_k\zeta)$$

$$s_k(\zeta) = \delta_k^2\left[\left(\frac{\sin\delta_k}{\delta_k} + \cos\delta_k\right)\cos\delta_k\zeta + \zeta\sin\delta_k\sin\delta_k\zeta\right]$$

The summation is carried out over the roots δ_k, having positive real part, since the roots with negative real part provide the same solution, in view of the fact that δ_k occurs as a square in Eq. (1.3).

We seek the state of stress of curl type in the form

$$u^{(3)} = a(\zeta)\ \partial F / \partial y, \qquad v^{(3)} = -a(\zeta)\partial F / \partial x, \qquad w^{(3)} \equiv 0$$

Arguing as before, we find

$$u^{(3)} = 2\nu\lambda\sum_{p=1}^{\infty}\cos\rho_p\zeta\frac{\partial F_p}{\partial\eta}, \qquad v^{(3)} = -2\nu\lambda\sum_{p=1}^{\infty}\cos\rho_p\zeta\frac{\partial F_p}{\partial\xi}, \qquad w^{(3)} \equiv 0$$

$$\sigma_x^{(3)} = -\sigma_y^{(3)} = \frac{2\mu}{a}2\nu\lambda\sum_{p=1}^{\infty}\cos\rho_p\zeta\frac{\partial^2 F_p}{\partial\xi\partial\eta}, \qquad \sigma_z^{(3)} = 0$$

$$\tau_{xy}^{(3)} = \frac{2\mu}{a}\nu\lambda\sum_{p=1}^{\infty}\cos\rho_p\zeta\left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2}\right)F_p$$

$$\tau_{xz}^{(3)} = -\frac{2\mu}{a}\nu\sum_{p=1}^{\infty}\rho_p\sin\rho_p\zeta\frac{\partial F_p}{\partial\eta}, \qquad \tau_{yz}^{(3)} = \frac{2\mu}{a}\nu\sum_{p=1}^{\infty}\rho_p\sin\rho_p\zeta\frac{\partial F_p}{\partial\xi} \qquad (1.7)$$

The functions F_p satisfy the same relations as the A_k

$$\partial^2 F_p / \partial\xi^2 + \partial^2 F_p / \partial\eta^2 - \rho_p^2 F_p / \lambda^2 = 0, \qquad \rho_p = p\pi \qquad (p = 1, 2, 3, \ldots)$$

These solutions were obtained in a different way in [2].

2. The problem of determination of the state of stress in a plate which has been formulated will be solved if the boundary conditions for the functions $\varphi(z)$, $\psi(z)$, A_k, F_p are determined in accordance with the tractions specified on Γ_2. In order to do this we shall use the principle of virtual work. We introduce a system of local dimensionless coordinates [3] related to the contour of the plate in plan, s, n, ζ (Fig. 2).

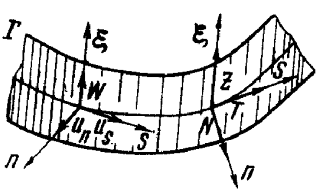

Fig. 2.

We shall consider that at each point of the surface Γ_2 the system of tractions $N(\zeta, s)$, $T(\zeta, s)$, $Z(\zeta, s)$ or the system $X_n^0(\zeta, s)$, $Y_n^0(\zeta, s)$, $Z_n^0(\zeta, s)$, is specified. These two systems are, of course, each expressible in terms of the other. The displacement components in the system of axes n, s are u_n and u_s. We shall seek the state of deformation of the plate in the form

$$u = u^{(1)} + u^{(2)} + u^{(3)}, \quad v = v^{(1)} + v^{(2)} + v^{(3)}, \quad w = w^{(1)} + w^{(2)} + w^{(3)} \qquad (2.1)$$

Then considering that the stresses of (2.1) are exact solutions of the Navier equations of the theory of elasticity, the virtual work equation may be taken in the form [4]

$$\iint_{\Gamma_2}[(X_n^{(1)} + X_n^{(2)} + X_n^{(3)})\delta u^{(1)} + (Y_n^{(1)} + Y_n^{(2)} + Y_n^{(3)})\delta v^{(1)} +$$
$$+ (Z_n^{(1)} + Z_n^{(2)} + Z_n^{(3)})\delta w^{(1)} + X_n^{(1)}\delta(u^{(2)} + u^{(3)}) + Y_n^{(1)}\delta(v^{(2)} +$$

$$+ v^{(3)}) + Z_n^{(1)}\delta (w^{(2)} + w^{(3)}) + (\sigma_n^{(2)} + \sigma_n^{(3)})\, \delta\, (u_n^{(2)} + u_n^{(3)}) + (\tau_{ns}^{(2)} + \tau_{ns}^{(3)})\delta\, (u_s^{(3)} + u_s^{(2)}) + (\tau_{nz}^{(2)} + \tau_{nz}^{(3)})\delta\, (w^{(2)} + w^{(3)})]d\sigma = \iint_{\Gamma_2} [X_n^0\delta u + Y_n^0\delta v + Z_n^0\delta w]d\sigma \tag{2.2}$$

In order to use this equation, it is necessary to transform to stresses and displacements in the local coordinates n and s. We obtain

$$\sigma_n^{(2)} + \sigma_n^{(3)} = \frac{2\mu}{a}\Big[\frac{\nu-1}{\lambda}\sum_{k=1}^{\infty} p_k(\zeta)\, A_k + \lambda \sum_{k=1}^{\infty} \alpha_k(\zeta)\,\frac{\partial^2 A_k}{\partial n^2} + 2\nu\lambda \sum_{p=1}^{\infty}\Big(\frac{1}{H}\frac{\partial^2 F_p}{\partial s\partial n} - \frac{a}{R}\frac{1}{H}\frac{\partial F_p}{\partial s}\Big)\cos\rho_p\zeta\Big] \tag{2.3}$$

$$\tau_{ns}^{(2)} + \tau_{ns}^{(3)} = \frac{2\mu}{a}\lambda\Big[\sum_{k=1}^{\infty}\alpha_k(\zeta)\Big(\frac{1}{H}\frac{\partial^2 A_k}{\partial s\partial n} - \frac{a}{R}\frac{1}{H^2}\frac{\partial A_k}{\partial s}\Big) + \nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\Big(\frac{1}{H^2}\frac{\partial^2 F_p}{\partial s^2} + \frac{a}{R}\frac{1}{H}\frac{\partial F_p}{\partial n} - \frac{\partial^2 F_p}{\partial n^2} + n\frac{aR'}{R^2}\frac{1}{H^3}\frac{\partial F_p}{\partial s}\Big)\Big]$$

$$\tau_{nz}^{(2)} + \tau_{nz}^{(3)} = \frac{2\mu}{a}\nu\Big[\sum_{k=1}^{\infty}\gamma_k(\zeta)\frac{\partial A_k}{\partial n} + \sum_{p=1}^{\infty}\rho_p\sin\rho_p\zeta\,\frac{1}{H}\frac{\partial F_p}{\partial s}\Big]$$

$$u_n^{(2)} + u_n^{(3)} = \lambda\Big[\sum_{k=1}^{\infty}\alpha_k(\zeta)\,\frac{\partial A_k}{\partial n} + 2\nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\,\frac{1}{H}\frac{\partial F_p}{\partial s}\Big]$$

$$u_s^{(2)} + u_s^{(3)} = \lambda\Big[\sum_{k=1}^{\infty}\alpha_k(\zeta)\,\frac{1}{H}\frac{\partial A_k}{\partial s} - 2\nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\,\frac{\partial F_p}{\partial n}\Big]$$

$$w^{(2)} + w^{(3)} = \sum_{k=1}^{\infty}\beta_k(\zeta)\, A_k \qquad \Big(H = 1 + n\frac{a}{R}\Big) \tag{2.4}$$

where R is the radius of curvature of the contour of the plate in plan.

In accordance with the well-known formulas, we can obtain from these equations

$X_n = \sigma_n l - \tau_{ns} m,\quad Y_n = \sigma_n m + \tau_{ns} l,\quad Z_n = \tau_{nz},$ where l and m are direction cosines.

The displacements $u^{(2)}$, $v^{(2)}$, $u^{(3)}$ and $v^{(3)}$, which occur in Eq. (2.2) can be written in local coordinates by using the expressions

$$u = u_n l - u_s m, \qquad v = u_n m + u_s l \tag{2.5}$$

For the solution of biharmonic type, we have

$$X_n^{(1)} + iY_n^{(1)} = -i\frac{\partial}{\partial s}(\varphi + z\bar{\varphi}' + \bar{\psi}) - \lambda^2 a^2\frac{\nu-1}{3\nu-1}\Big(\frac{1}{3} - \zeta^2\Big)i\frac{\partial}{\partial s}\bar{\varphi}'',\quad Z_n^{(1)} = 0$$

We write the values of the tractions $X_n^{(1)}$ and $Y_n^{(1)}$ on the boundary in the form

$$X_n^{(1)} = R_{1x}(\varphi, \psi) + \lambda^2\zeta^2 R_{2x}(\varphi),\quad Y_n^{(1)} = R_{1y}(\varphi, \psi) + \lambda^2\zeta^2 R_{2y}(\varphi) \tag{2.6}$$

Here R_{1x}, R_{2x}, R_{1y} and R_{2y} are operators the meaning of which becomes clear from examination of Eq. (2.5).

We introduce further the operators S_{1k} and S_{2k}, which are defined as follows: let $a_k(s)$ be the boundary values of the function $A_k(s, n)$, which satisfies (1.3). Then

$$S_{1k}a_k = \lambda\frac{\partial A_k}{\partial n}\Big|_{\Gamma}, \qquad S_{2k}a_k = \lambda^2\frac{\partial^2 A_k}{\partial n^2}\Big|_{\Gamma} \tag{2.7}$$

Analogously we have

$$S_{1p}f_p = \lambda \frac{\partial F_p}{\partial n}\bigg|_\Gamma, \qquad S_{2p}f_p = \lambda^2 \frac{\partial^2 F_p}{\partial n^2}\bigg|_\Gamma, \qquad f_p = F_p(s,n)|_\Gamma \tag{2.8}$$

For the stresses and displacements on the boundary in the solutions of potential and curl type, we obtain

$$\frac{a}{2\mu} X_n^{(2)} = \frac{1}{\lambda} l\left[(\nu-1)\sum_{k=1}^{\infty} p_k(\zeta)a_k + \sum_{k=1}^{\infty}\alpha_k(\zeta)S_{2k}a_k\right] - \\ - m\sum_{k=1}^{\infty}\alpha_k(\zeta)\left(\frac{\partial}{\partial s}S_{1k}a_k - \lambda\frac{a}{R}a_k'\right)$$

$$\frac{a}{2\mu} Y_n^{(2)} = \frac{1}{\lambda} m\sum_{k=1}^{\infty}[(\nu-1)p_k(\zeta)a_k + \alpha_k(\zeta)S_{2k}a_k] + \\ + l\sum_{k=1}^{\infty}\alpha_k(\zeta)\left(\frac{a}{\partial s}S_{1k}a_k - \lambda\frac{a}{R}a_k'\right)$$

$$\frac{a}{2\mu} Z_n^{(2)} = \frac{a}{2\mu}\tau_{nz}^{(2)} = \frac{1}{\lambda}\nu\sum_{k=1}^{\infty}\gamma_k(\zeta)S_{1k}a_k,$$

$$\frac{a}{2\mu}\sigma_n^{(2)} = \frac{1}{\lambda}\left[(\nu-1)\sum_{k=1}^{\infty}p_k(\zeta)a_k + \sum_{k=1}^{\infty}\alpha_k(\zeta)S_{2k}a_k\right]$$

$$\frac{a}{2\mu}\tau_{ns}^{(2)} = \sum_{k=1}^{\infty}\alpha_k(\zeta)\left(\frac{\partial}{\partial s}S_{1k}a_k - \lambda\frac{a}{R}a_k'\right) \tag{2,9}$$

$$u_n^{(2)} = \sum_{k=1}^{\infty}\alpha_k(\zeta)S_{1k}a_k, \qquad u^{(2)} = \sum_{k=1}^{\infty}\alpha_k(\zeta)(lS_{1k}a_k - \lambda m a_k')$$

$$u_s^{(2)} = \lambda\sum_{k=1}^{\infty}\alpha_k(\zeta)a_k', \quad v^{(2)} = \sum_{k=1}^{\infty}\alpha_k(\zeta)(mS_{1k}a_k + \lambda l a_k') \quad w^{(2)} = \sum_{k=1}^{\infty}\beta_k(\zeta)a_k$$

$$\frac{a}{2\mu} X_n^{(3)} = \nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\left[\frac{1}{\lambda}mS_{2p}f_p + \left(2l\frac{\partial}{\partial s} - m\frac{a}{R}\right)S_{1p}f_p - \\ - \lambda\left(2l\frac{a}{R}f_p' - mf_p''\right)\right]$$

$$\frac{a}{2\mu} Y_n^{(3)} = \nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\left[-\frac{1}{\lambda}lS_{2p}f_p + \left(2m\frac{\partial}{\partial s} + l\frac{a}{R}\right)S_{1p}f_p - \\ - \lambda\left(2m\frac{a}{R}f_p' - lf_p''\right)\right]$$

$$\frac{a}{2\mu} Z_n^{(3)} = \frac{a}{2\mu}\tau_{nz}^{(3)} = \nu\sum_{p=1}^{\infty}\rho_p\sin\rho_p\zeta f_p'$$

$$\frac{a}{2\mu}\sigma_n^{(3)} = 2\nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\left(\frac{\partial}{\partial s}S_{1p}f_p - \lambda\frac{a}{R}f_p'\right)$$

$$\frac{a}{2\mu}\tau_{ns}^{(3)} = \nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\left(-\frac{1}{\lambda}S_{2p}f_p + \frac{a}{R}S_{1p}f_p + \lambda f_p''\right) \tag{2.10}$$

$$u_n^{(3)} = 2\nu\lambda \sum_{p=1}^{\infty} \cos \rho_p\zeta f_p', \qquad u^{(3)} = 2\nu \sum_{p=1}^{\infty} \cos \rho_p\zeta (\lambda l f_p' + mS_{1p}f_p)$$

$$u_s^{(3)} = -2\nu \sum_{p=1}^{\infty} \cos \rho_p\zeta S_{1p}f_p, \qquad v^{(3)} = 2\nu \sum_{p=1}^{\infty} \cos \rho_p\zeta (\lambda m f_p' - lS_{1p}f_p)$$

$$w^{(3)} \equiv 0$$

In order to make use of the variational relation (2.2), the variations in the solution of biharmonic type must be expressed in terms of any two independent ones. The state of stress of biharmonic type is described by the fuctions $\varphi(z)$ and $\Psi(z)$ and would be completely determined if the displacements u and v corresponding to the given solution of this type were known on the contour. It is, therefore natural to express the variations of all quantities related to this solution in terms of the variations δu and δv on the contour. For $\delta w^{(1)}$ for instance, we have

$$\delta w^{(1)} = \zeta\lambda (K_1\delta u^{(1)} + K_2\delta v^{(1)}) \tag{2.11}$$

where K_1 and K_2 are certain integro-differential operators defined by the two-dimensional problem of the theory of elasticity in terms of displacements. As other independent variations we shall take the variations of the boundary values of A_k and F_p.

Varying only the boundary value of u, we obtain from Eq. (2.2)

$$2R_{1x}(\varphi, \psi) + \frac{2}{3}\lambda^2 R_{2x}(\varphi) + \frac{2\mu}{a}\sum_{k=1}^{\infty} J_{k1}\Big[l\frac{1}{\lambda}(S_{2k}a_k - \delta_k^2 a_k) -$$

$$- m\left(\frac{\partial}{\partial s} S_{1k}a_k - \lambda\frac{a}{R}a_k'\right)\Big] + \frac{2\mu}{a}\nu K_1^*\left(\sum_{k=1}^{\infty} J_{k2}S_{1k}a_k + \lambda\sum_{p=1}^{\infty}\rho_p J_{p3}f_p'\right) =$$

$$= 2\langle X_n^\circ\rangle + 2\lambda K_1^*\langle Z_n^\circ\rangle \tag{2.12}$$

Here and in what follows angle brackets denote mean values of quantities. Varying v on the boundary, we find the analogous relations

$$2R_{1y}(\varphi, \psi) + \frac{2}{3}\lambda^2 R_{2y}(\varphi) + \frac{2\mu}{a}\sum_{k=1}^{\infty} J_{k1}\Big[m\frac{1}{\lambda}(S_{2k}a_k - \delta_k^2 a_k) +$$

$$+ l\left(\frac{\partial}{\partial s} S_k a_k - \lambda\frac{a}{R}a_k'\right)\Big] + \frac{2\mu}{a}\nu K_2^*\left(\sum_{k=1}^{\infty} J_{k2}S_{1k}a_k + \lambda\sum_{p=1}^{\infty}\rho_p J_{p3}f_p'\right) =$$

$$= 2\langle Y_n^\circ\rangle + 2\lambda K_2^*\langle Z_n^\circ\rangle \tag{2.13}$$

where

$$(\nu - 1)\int_{-1}^{1} p_k(\zeta)\,d\zeta = -\delta_k^2 J_{k1}, \qquad \int_{-1}^{1}\zeta\gamma_k(\zeta)\,d\zeta = J_{k2}, \qquad \int_{-1}^{1}\zeta\sin\rho_p\zeta\,d\zeta = J_p^3$$

$$\int_{-1}^{1}\alpha_k(\zeta)\,d\zeta = J_{k1}, \quad \int_{-1}^{1}\cos\rho_p\zeta\,d\zeta = 0, \quad {}^1/_2\int_{-1}^{1}X_n^\circ d\zeta = \langle X_n^\circ\rangle, \quad {}^1/_2\int_{-1}^{1}Y_n^\circ d\zeta = \langle Y_n^\circ\rangle$$

Here $K_1{}^*$ and $K_2{}^*$ are the operators which are adjoint to K_1 and K_2. We recall that an operator K_1 is related to its adjoint $K_1{}^*$ by the following identity valid for any functions $f_1(s)$ and $f_2(s)$ on the contour

$$\oint f_1(s) K_i f_2(s) ds = \oint f_2(s) K_i{}^* f_1(s)\, ds$$

Varying the boundary value of A_k, we obtain the infinite system

$$J_{m1}\Big[S_{1m}^* l R_{1x}(\varphi, \psi) + \lambda \frac{d}{ds}(m R_{1x}(\varphi, \psi))\Big] +$$

$$+ \lambda^2 J_{m4}\Big[S_{1m}^* l R_{2x}(\varphi) + \lambda \frac{d}{ds} m R_{2x}(\varphi)\Big] + J_{m1}\Big[S_{1m}^* m R_{1y}(\varphi, \psi) -$$

$$- \lambda \frac{d}{ds}(l R_{1y}(\varphi, \psi))\Big] + \lambda^2 J_{m4}\Big[S_{1m}^* m R_{2y}(\varphi) - \lambda \frac{d}{ds}(l R_{2y}(\varphi))\Big] +$$

$$+ \frac{2\mu}{a} S_{1m}^* \Big[\frac{1}{\lambda}\sum_{k=1}^{\infty}((\nu-1) J_{km5}\, a_k + J_{km6} S_{2k} a_k) + 2\nu \sum_{p=1}^{\infty} J_{pm7}\Big(\frac{d}{ds} S_{1p} f_p - \lambda \frac{a}{R} f_p{}'\Big)\Big] -$$

$$- \frac{2\mu}{a}\frac{d}{ds}\Big[\lambda \sum_{k=1}^{\infty} J_{km6}\Big(\frac{d}{ds} S_{1k} a_k - \lambda \frac{a}{R} a_k{}'\Big) +$$

$$+ \nu \sum_{p=1}^{\infty} J_{pm7}\Big(- S_{2p} f_p + \lambda \frac{a}{R} S_{1p} f_p + \lambda^2 f_p{}''\Big)\Big] + \frac{2\mu}{a}\nu\Big[\frac{1}{\lambda}\sum_{k=1}^{\infty} J_{km8} S_{1k} a_k +$$

$$+ \sum_{p=1}^{\infty} \rho_p J_{pm9} f_p{}'\Big] = S_{1m}^* N_m - \lambda \frac{d}{ds} T_m + Z_m \quad (m = 1, 2, 3, \ldots) \qquad (2.14)$$

where S_{1m}^* is adjoint to the operator S_{1m} and

$$\int_{-1}^{1} \zeta^2 \alpha_m(\zeta)\, d\zeta = J_{m4}, \qquad \int_{-1}^{1} p_k(\zeta)\, \alpha_m(\zeta)\, d\zeta = J_{km5}, \qquad \int_{-1}^{1} N \alpha_m(\zeta)\, d\zeta = N_m,$$

$$\int_{-1}^{1} \alpha_k(\zeta)\, \alpha_m(\zeta)\, d\zeta = J_{km6}, \qquad \int_{-1}^{1} \alpha_m(\zeta) \cos \rho_p \zeta\, d\zeta = J_{pm7}, \qquad \int_{-1}^{1} T \alpha_m(\zeta)\, d\zeta = T_m,$$

$$\int_{-1}^{1} \gamma_k(\zeta)\, \beta_m(\zeta)\, d\zeta = J_{km8}, \qquad \int_{-1}^{1} \beta_m(\zeta) \sin \rho_p \zeta\, d\zeta = J_{pm9}, \qquad \int_{-1}^{1} Z \beta_m(\zeta)\, d\zeta = Z_m$$

Finally, by varying the boundary value of the function F_p, we have

$$- \lambda^3 J_{t10} \frac{d}{ds}(l R_{2x}(\varphi) + m R_{2y}(\varphi)) + \lambda^2 J_{t10} S_{1t}{}^* (m R_{2x}(\varphi) - l R_{2y}(\varphi)) -$$

$$- \frac{2\mu}{a}\frac{d}{ds}\sum_{k=1}^{\infty}[(\nu-1) J_{kt11} a_k + J_{kt7} S_{2k} a_k] + \qquad (t = 1, 2, 3, \ldots) \qquad (2.15)$$

$$+ \frac{2\mu}{a} S_{1t}{}^* \sum_{k=1}^{\infty} J_{kt7}\Big(\frac{d}{ds} S_{1k} a_k - \lambda \frac{a}{R} a_k{}'\Big) - \frac{2\mu}{a}\nu\Big[-\frac{1}{\lambda} S_{1t}{}^* S_{2t} f_t +$$

$$+ S_{1t}{}^* \frac{a}{R} S_{1t} f_t + 2\frac{d^2}{ds^2} S_{1t} f_t - \lambda \frac{d}{ds}\Big(\frac{a}{R} f_t{}'\Big) + \lambda S_{1t}{}^* f_t{}''\Big] = - \lambda \frac{dN_t}{ds} - S_{1t}{}^* T_t$$

where

$$\int_{-1}^{1} \zeta^2 \cos \rho_t \zeta \, d\zeta = J_{t10}, \quad \int_{-1}^{1} p_k(\zeta) \cos \rho_t \zeta \, d\zeta = J_{kt11}, \quad \int_{-1}^{1} \cos \rho_p \zeta \cos \rho_t \zeta \, d\zeta = 0$$

$$\int_{-1}^{1} N \cos \rho_t \zeta \, d\zeta = N_t, \quad \int_{-1}^{1} T \cos \rho_t \zeta \, d\zeta = T_t, \quad \int_{-1}^{1} \cos^2 \rho_t \zeta \, d\zeta = 1$$

Thus a complete system of equations has been obtained which determines the solution of the problem in its entirety. Eq. (2.15) serves to determine the f_t for $t = 1, 2, 3, \ldots$. All the a_k, are found from the system (2.14). Finally, Eqs. (2.12) and (2.13) provide the boundary conditions for the functions $\varphi(z)$ and $\Psi(z)$. If, however, the function f_p and a_k, are eliminated from (2.12) and (2.13) with the aid of (2.15) and (2.14), then we directly obtain the boundary conditions for the functions φ and Ψ of the solution of biharmonic type.

3. The construction of the operators S_{1k} and S_{2k} can be carried out effectively for small values of the parameter λ by using asymptotic expansions of the functions A_k and F_p, having the following form

$$A_k(s, n) = \left\{ a_k(s) - \frac{a}{2R} n a_k + \frac{1}{2\delta_k} \left[\frac{3a^2\delta_k}{4R^2} n^2 a_k - \frac{a^2}{4R^2} \lambda n a_k - \lambda n a_k'' \right] + \right.$$
$$+ \frac{1}{2\delta_k} \left[-\frac{5a^3}{8R^3} \delta_k n^3 a_k + \lambda n^2 \left(\frac{3a}{2R} a_k'' - \frac{aR'}{R^2} a_k' + a \frac{2R'^2 - RR'' + {}^3/_2 a^2}{4R^3} a_k \right) + \right.$$
$$\left. \left. + \frac{\lambda^2 n}{2\delta_k} \left(-\frac{2a}{R} a_k'' + \frac{2aR'}{R^2} a_k' - a \frac{2R'^2 - RR'' + a^2}{2R^3} a_k \right) \right] + \ldots \right\} \exp \frac{\delta_k n}{\lambda} \tag{3.1}$$

Analogous representations can also be written out for $F_p(s, n)$. As a point moves into the interior of the region (as $n \to -\infty$), the solution decays exponentially. The operators S_{1k} and S_{2k} introduced earlier are then given by the relations

$$S_{1k} = \delta_k - \frac{a}{2R} \lambda - \frac{1}{2\delta_k} \lambda^2 \left(\frac{a^2}{R^2} + \frac{d^2}{ds^2} \right) + \frac{1}{2\delta_k^2} \frac{a}{R} \lambda^3 \left(-\frac{d^2}{ds^2} + \frac{R'}{R} \frac{d}{ds} - \right.$$
$$\left. - \frac{2R'^2 - RR'' + a^2}{4R^2} \right) + \ldots \tag{3.2}$$

$$S_{2k} = \delta_k^2 - \frac{a}{R} \delta_k \lambda + \lambda^2 \left(\frac{a^2}{2R^2} - \frac{d^2}{ds^2} \right) + \frac{a}{2R\delta_k} \lambda^3 \left(\frac{d^2}{ds^2} + \frac{a^2}{4R^2} \right) + \ldots \tag{3.3}$$

The corresponding adjoint operator is

$$S_{1k}^* = \delta_k^2 - \frac{a}{2R} \lambda - \frac{1}{2\delta_k} \lambda^2 \left(\frac{a^2}{R^2} + \frac{d^2}{ds^2} \right) +$$
$$+ \frac{1}{2R\delta_k^2} \lambda^3 \left(-\frac{d^2}{ds^2} - \frac{R'}{R} \frac{d}{ds} - \frac{2R'^2 - RR'' + a^2}{4R^2} \right) + \ldots \tag{3.4}$$

We shall seek the solution in the form

$$a_k(s) = a_{k0} + \lambda a_{k1} + \lambda^2 a_{k2} + \ldots, \quad \varphi(z) = \varphi_0 + \lambda \varphi_1 + \lambda^2 \varphi_2 + \ldots \tag{3.5}$$
$$f_p(s) = f_{p0} + \lambda f_{p1} + \lambda^2 f_{p2} + \ldots, \quad \psi(z) = \psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \ldots$$

assuming that the tractions specified on the contour are also representable in series form [3]

$$N = \lambda N_1 + \lambda^2 N_2 + \ldots, \quad T = \lambda T_1 + \lambda^2 T_2 + \ldots, \quad Z = \lambda^2 Z_2 + \ldots$$
$$X_n^\circ = \lambda X_{n1}^\circ + \lambda^2 X_{n2}^\circ + \ldots, Y_n^\circ = \lambda Y_{n1}^\circ + \lambda^2 Y_{n2}^\circ + \ldots, Z_n^\circ = \lambda^2 Z_{n2}^\circ$$

Using these expansions, we now collect terms having the same power of λ in Eqs. (2.12) to (2.15). For all the functions which have been introduced, we thus obtain the boundary data for each approximation in powers of λ separately. For the lowest power of λ, i.e., in the zeroth approximation

$$2R_{1x}(\varphi_0, \psi_0) - \frac{2\mu}{a}\sum_{k=1}^{\infty} J_{k1}\delta_k\left(\frac{a}{R}\, la_{k0} + ma_{k0}'\right) + \nu\frac{2\mu}{a}K_1^*\sum_{k=1}^{\infty} J_{k2}\delta_k a_{k0} = 0 \tag{3.6}$$

$$2R_{1y}(\varphi_0, \psi_0) - \frac{2\mu}{a}\sum_{k=1}^{\infty} J_{k1}\delta_k\left(\frac{a}{R}\, ma_{k0} - la_{k0}'\right) + \nu\frac{2\mu}{a}K_2^*\sum_{k=1}^{\infty} J_{k2}\delta_k a_{k0} = 0 \tag{3.7}$$

$$\frac{2\mu}{a}\sum_{k=1}^{\infty}\left[(\nu-1)\delta_m J_{km5}a_{k0} + \delta_m J_{km6}\delta_k^2 a_{k0} + \nu J_{km8}\delta_k a_{k0}\right] = 0 \tag{3.8}$$

$$\frac{2\mu}{a}\nu\rho_t^2 f_{t0} = 0 \tag{3.9}$$

It is immediately apparent from this that $f_{t0} = 0$. The system (3.8) is such that $a_{k0} = 0$. Then we obtain from (3.6) and (3.7) that $\varphi_0 = \psi_0 = 0$.

For the next higher power of λ, we have in the first approximation

$$2R_{1x}(\varphi_1, \psi_1) - \frac{2\mu}{a}\sum_{k=1}^{\infty} J_{k1}\delta_k\left(\frac{a}{R}\, la_{k1} + ma_{k1}'\right) + \nu\frac{2\mu}{a}K_1^*\sum_{k=1}^{\infty} J_{k2}\delta_k a_{k1} = \langle X_n^\circ\rangle \tag{3.10}$$

$$2R_{1y}(\varphi_1, \psi_1) - \frac{2\mu}{a}\sum_{k=1}^{\infty} J_{k1}\delta_k\left(\frac{a}{R}\, ma_{k1} - la_{k1}'\right) + \nu\frac{2\mu}{a}K_2^*\sum_{k=1}^{\infty} J_{k2}\delta_k a_{k1} = \langle Y_n^\circ\rangle \tag{3.11}$$

$$\frac{2\mu}{a}\sum_{k=1}^{\infty}\left[\nu-1)\delta_m J_{km5}a_{k1} + \delta_m J_{km6}\delta_k^2 a_{k1} + \nu I_{km8}\delta_k a_{k1}\right] = 0 \tag{3.12}$$

$$\frac{2\mu}{a}\nu\rho_t^2 f_{t1} = 0 \tag{3.13}$$

It is clear from this that $f_{t1} = a_{k1} = 0$. From (3.10) and (3.11) we obtain the boundary conditions for $\varphi_1(z)$ and $\psi_1(z)$:

$$\frac{d}{ds}\left(\varphi_1 + z\overline{\varphi_1}' + \overline{\psi}_1\right) = i\left(\langle \mathring{X}_{n1}\rangle + i\langle \mathring{Y}_{n1}\rangle\right) \tag{3.14}$$

which coincides exactly with the Kolosov-Muskhelishvili boundary condition for $\varphi(z)$ and $\psi(z)$ in the two-dimensional theory of elasticity.

In the second approximation we obtain

$$\frac{d}{ds}\left(\varphi_2 + z\overline{\varphi}_2' + \overline{\psi}_2\right) = i\left(\langle \mathring{X}_{n2}\rangle + i\langle \mathring{Y}_{n2}\rangle\right) + \tag{3.15}$$
$$+ \frac{2\mu}{a}\frac{a}{R}(1-\nu)(li - m)\sum_{k=1}^{\infty}\frac{\sin^2\delta_k}{\delta_k}a_{k2} +$$
$$+ \frac{2\mu}{a}(1-\nu)(l + mi)\sum_{k=1}^{\infty}\frac{\sin^2\delta_k}{\delta_k}a_{k2}' - \frac{2\mu}{a}2\nu\sum_{k=1}^{\infty}\frac{\sin^2\delta_k}{\delta_k}(iK_1^* - K_2^*)a_{k2}$$

$(m = 1, 2, \ldots)$

$$-4\nu\sum_{\substack{k=1\\k\neq m}}^{\infty}\frac{\delta_k^2\delta_m^2(\sin^2\delta_m-\sin^2\delta_k)}{(\delta_m^2-\delta_k^2)(\delta_m-\delta_k)}[(\nu-1)(\delta_k^2+\delta_m^2)+2(\nu+1)\delta_k\delta_m]a_{k2}+$$

$$+2\nu^2\delta_m^3\left(\frac{2}{3}\sin^2\delta_m-1\right)a_{m2}=-\frac{a}{2\mu}\frac{\sin^2\delta_m}{\delta_m}\left[li\frac{d}{ds}(\varphi_1-\overline{\varphi}_1+z\overline{\varphi}_1'-\overline{z}\varphi_1'+\right.$$

$$\left.+\overline{\psi}_1-\psi_1)+m\frac{d}{ds}(\varphi_1+\overline{\varphi}_1+\overline{z}\varphi_1'+z\overline{\varphi}_1'+\overline{\psi}_1+\psi_1)\right]-\frac{a}{2\mu}\delta_m N_{m1}\quad(3.16)$$

In this approximation

$$f_{t2}=-\frac{a}{2\mu\nu}\frac{T_{t1}}{\rho_t^2}\qquad(t=1,2,3,\ldots)\qquad(3.17)$$

The matrix of the infinite system of Eqs. (3.16) depends neither on the load nor on the region occupied by the plate, but is affected only by Poisson's ratio. The form of the matrix is the same as in the case of bending of a plate: the difference consists in the values of the numbers δ_k. The system can be solved well by the method of truncation. The same constructions are carried out for the third and higher approximations. At each stage of approximation it is necessary to solve a two-dimensional Kolosov-Muskhelishvili problem for the given region but for a different right-hand side. Moreover, the a_k must be determined from an infinite system of equations having the same matrix for all the approximations.

4. Thus, we have for the boundary values of the functions which have been introduced

$$\varphi(z)=\lambda\varphi_1+\lambda^2\varphi_2+\ldots,\qquad\psi(z)=\lambda\psi_1+\lambda^2\psi_2+\ldots$$

$$a_k(s)=\lambda^2a_{k2}+\ldots,\qquad f_p(s)=\lambda^2f_{p2}+\ldots\qquad(4.1)$$

From this we finally obtain the following expressions for the stresses and displacements

$$\sigma_n=-\lambda\frac{1}{2}\left[li\frac{\partial}{\partial s}(\varphi_1-\overline{\varphi}_1+z\overline{\varphi}_1'-\overline{z}\varphi_1'+\overline{\psi}_1-\psi_1)+\right.$$

$$\left.+m\frac{\partial}{\partial s}(\varphi_1+\overline{\varphi}_1+\overline{z}\varphi_1'+z\overline{\varphi}_1'+\psi_1+\overline{\psi}_1)\right]+$$

$$+\frac{2\mu}{a}\lambda\sum_{k=1}^{\infty}\left(1-\frac{a}{2R}n+\frac{3a^2}{8R^2}n^2-\ldots\right)[(\nu-1)p_k(\zeta)+\delta_k^2\alpha_k(\zeta)]a_{k2}(s)\exp\frac{\delta_kn}{\lambda}-$$

$$-\lambda^2\frac{1}{2}\left[li\frac{\partial}{\partial s}(\varphi_2-\overline{\varphi}_2+z\overline{\varphi}_2'-\overline{z}\varphi_2'+\overline{\psi}_2-\psi_2)+\right.$$

$$\left.+m\frac{\partial}{\partial s}(\varphi_2+\overline{\varphi}_2+z\varphi_2'+z\overline{\varphi}_2'+\psi_2+\overline{\psi}_2)\right]+$$

$$+\frac{2\mu}{a}\lambda^2 2\nu\sum_{p=1}^{\infty}\frac{1}{H}\cos\rho_p\zeta\rho_p\frac{\partial}{\partial s}\left[f_{p2}\left(1-\frac{a}{2R}n+\frac{3a^2}{8R^2}n^2+\ldots\right)\right]\exp\frac{\rho_pn}{\lambda}+$$

$$+\frac{2\mu}{a}\lambda^2\sum_{k=1}^{\infty}\left\{\left[(\nu-1)p_k(\zeta)+\delta_k^2\alpha_k(\zeta)\right]\left(1-\frac{1}{2R}n+\frac{3a^2}{8R^2}n^2+\ldots\right)a_{k3}(s)+\right.$$

$$+\left[\frac{\nu-1}{2\delta_k}p_k(\zeta)+\frac{\delta_k}{2}\alpha_k(\zeta)\right]\left(-\frac{a^2}{4R^2}n-n\frac{\partial^2}{\partial s^2}-\ldots\right)a_{k2}(s)+$$

$$\left.+2(\nu-1)\delta_kp_k(\zeta)\left(-\frac{a}{2R}+\frac{3a^2}{4R^2}n+\ldots\right)a_{k2}(s)\right\}\exp\frac{\delta_kn}{\lambda}+\ldots\quad(4.2)$$

$$\tau_{ns} = \lambda \frac{1}{2}\Big[im \frac{\partial}{\partial s}(\varphi_1 - \bar{\varphi}_1 + z\bar{\varphi}_1' - \bar{z}\varphi_1' + \bar{\psi}_1 - \psi_1) - l \frac{\partial}{\partial s}(\varphi_1 + \bar{\varphi}_1 + z\bar{\varphi}_1' +$$
$$+ \bar{z}\varphi_1' + \psi_1 + \bar{\psi}_1)\Big] - \frac{2\mu}{a}\lambda\nu \sum_{p=1}^{\infty} \cos\rho_p\zeta\rho_p^2\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 + \ldots\Big) \times$$
$$\times f_{p2}(s)\exp\frac{\rho_p n}{\lambda} + \lambda^2 \frac{1}{2}\Big[im \frac{\partial}{\partial s}(\varphi_2 - \bar{\varphi}_2 + z\bar{\varphi}_2' - \bar{z}\varphi_2' + \bar{\psi}_2 - \psi_2) -$$
$$- l\frac{\partial}{\partial s}(\varphi_2 + \bar{\varphi}_2 + z\bar{\varphi}_2' + \bar{z}\varphi_2' + \psi_2 + \bar{\psi}_2)\Big] + \frac{2\mu}{a}\lambda^2\Big\{\sum_{k=1}^{\infty}\alpha_k(\zeta)\frac{1}{H}\delta_k \times$$
$$\times \frac{\partial}{\partial s}\Big[\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 + \ldots\Big)a_{k2}(s)\Big]\exp\frac{\delta_k n}{\lambda} +$$
$$+ \nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\Big[\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 - \ldots\Big)\Big(\frac{a}{R}\frac{1}{H}f_{p2} - \rho_p^2 f_{p3}\Big) +$$
$$+ \rho_p\Big(\frac{a}{R} - \frac{a^2}{8R^2}n - \frac{1}{2}n\frac{\partial^2}{\partial s^2} + \frac{3a^2}{4R^2}n^2 + \ldots\Big)f_{p2}(s)\Big]\exp\frac{\rho_p n}{\lambda}\Big\} + \ldots \quad (4.3)$$

$$\tau_{nz} = \frac{2\mu\nu}{a}\lambda\sum_{k=1}^{\infty}\gamma_k(\zeta)\delta_k\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 + \ldots\Big)a_{k3}(s)\exp\frac{\delta_k n}{\lambda} +$$
$$+ \frac{2\mu\nu}{a}\lambda^2\Big\{\sum_{k=1}^{\infty}\gamma_k(\zeta)\Big[\delta_k\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 + \ldots\Big)a_{k3}(s) +$$
$$+ \Big(-\frac{a}{2R} + n\frac{5a^2}{8R} - \frac{1}{2}n\frac{\partial^2}{\partial s^2} + \ldots\Big)a_{k2}(s)\Big]\exp\frac{\delta_k n}{\lambda} -$$
$$- \sum_{p=1}^{\infty}\rho_p\sin\rho_p\zeta\frac{1}{H}\frac{\partial}{\partial s}\Big[\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 - \ldots\Big)f_{p2}(s)\Big]\exp\frac{\rho_p n}{\lambda} + \ldots \quad (4.4)$$

$$u_n = \frac{1}{4\mu}\lambda\{l[\varkappa(\varphi_1 + \bar{\varphi}_1) - z\bar{\varphi}_1' - \bar{z}\varphi_1' - \bar{\psi}_1 - \psi_1] - mi[\varkappa(\varphi_1 - \bar{\varphi}_1) -$$
$$- z\bar{\varphi}_1' + \bar{z}\varphi_1' - \bar{\psi}_1 + \psi_1]\} + \frac{1}{4\mu}\lambda^2\{l[\varkappa(\varphi_2 + \bar{\varphi}_2) - z\bar{\varphi}_2' - \bar{z}\varphi_2' - \bar{\psi}_2 - \psi_2] -$$
$$- mi[\varkappa(\varphi_2 - \bar{\varphi}_2) - z\bar{\varphi}_2' + \bar{z}\varphi_2' - \bar{\psi}_2 + \psi_2]\} +$$
$$+ \lambda^2\sum_{k=1}^{\infty}\alpha_k(\zeta)\delta_k\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 - \ldots\Big)a_{k2}(s)\exp\frac{\delta_k n}{\lambda} + \ldots$$

$$u_s = -\frac{1}{4\mu}\lambda\{m[\varkappa(\varphi_1 + \bar{\varphi}_1) - z\bar{\varphi}_1' - \bar{z}\varphi_1' - \bar{\psi}_1 - \psi_1] + li[\varkappa(\varphi_1 - \bar{\varphi}_1) -$$
$$- z\bar{\varphi}_1' + \bar{z}\psi_1' - \bar{\psi}_1 - \psi_1]\} - \frac{1}{4\mu}\lambda^2\{m[\varkappa(\varphi_2 + \bar{\varphi}_2) - z\bar{\varphi}_2' - \bar{z}\varphi_2' -$$
$$- \bar{\varphi}_2 - \psi_2] + li[\varkappa(\varphi_2 - \bar{\varphi}_2) - z\bar{\varphi}_2' + \bar{z}\varphi_2' - \bar{\psi}_2 + \psi_2]\} -$$
$$- 2\nu\lambda^2\sum_{p=1}^{\infty}\cos\rho_p\zeta\rho_p\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 + \ldots\Big)f_{p2}(s)\exp\frac{\rho_p n}{\lambda} + \ldots \quad (4.5)$$

$$w = -\frac{1}{\mu}\lambda\frac{\nu - 1}{3\nu - 1}a\zeta(\varphi_1' + \bar{\varphi}_1') - \frac{1}{\mu}\lambda^2\frac{\nu - 1}{3\nu - 1}a\zeta(\varphi_2' + \bar{\varphi}_2') +$$
$$+ \lambda^2\sum_{k=1}^{\infty}\beta_k(\zeta)\Big(1 - \frac{a}{2R}n + \frac{3a^2}{8R^2}n^2 + \ldots\Big)a_{k2}(s)\exp\frac{\delta_k n}{\lambda} + \ldots$$

As we move towards the interior of the region, the state of stress tends to one of biharmonic type, but to one which is not described by the boundary conditions of the two-dimensional state of stress. This latter is obtained as the first term in the expansion in powers of λ. St. Venant's principle as usually stated is not satisfied, since even far from the boundary the state of stress does not coincide with the two-dimensional one.

In Eqs. (4.5) for the displacements the first terms on the right-hand side correspond to the solution of the two-dimensional problem of the theory of elasticity. It follows from this that the error in the determination of the displacements in the two-dimensional problem will be of order λ compared with unity.

Let us examine the values of the stresses on the boundary $n=0$

$$\sigma_n = -\frac{\lambda}{2}\frac{d}{ds}[li(\varphi_1 - \bar{\varphi}_1 + z\bar{\varphi}_1' - \bar{z}\varphi_1' + \bar{\psi}_1 - \psi_1) + m(\varphi_1 + \bar{\varphi}_1 + \bar{z}\varphi_1' + z\bar{\varphi}_1' + \bar{\psi}_1 + \psi_1)]|_{n=0} + \frac{2\mu}{a}\lambda\sum_{k=1}^{\infty}[(\nu-1)p_k(\zeta) + \delta_k^2\alpha_k(\zeta)]a_{k2}(\zeta) - \frac{\lambda^2}{2}\frac{d}{ds}[li(\varphi_2 - \bar{\varphi}_2 + z\bar{\varphi}_2' - \bar{z}\varphi_2' + \bar{\psi}_2 - \psi_2) + m(\varphi_2 + \bar{\varphi}_2 + \bar{z}\varphi_2' + z\bar{\varphi}_2' + \bar{\psi}_2 + \psi_2)]|_{n=0} + \frac{2\mu}{a}\lambda^2\left\{2\nu\sum_{p=1}^{\infty}\rho_p\cos\rho_p\zeta f_{p2} + \sum_{k=1}^{\infty}[(\nu-1)p_k(\zeta) + \delta_k^2\alpha_k(\zeta)]a_{k3}(s) - (\nu-1)\frac{a}{R}\sum_{k=1}^{\infty}\delta_k p_k(\zeta)a_{k2}(s)\right\} + \dots \quad (4.6)$$

$$\tau_{ns} = \frac{\lambda}{2}\frac{d}{ds}[mi(\varphi_1 - \bar{\varphi}_1 + z\bar{\varphi}_1' - \bar{z}\varphi_1' + \bar{\psi}_1 - \psi_1) - l(\varphi_1 + \bar{\varphi}_1 + \bar{z}\varphi_1' + z\bar{\varphi}_1' + \bar{\psi}_1 + \psi_1)]|_{n=0} - \frac{2\mu}{a}\lambda\nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\rho_p^2 f_{p2}(s) + \frac{\lambda^2}{2}\frac{d}{ds}[mi(\varphi_2 - \bar{\varphi}_2 + z\bar{\varphi}_2' - \bar{z}\varphi_2' + \bar{\psi}_2 - \psi_2) - l(\varphi_2 + \bar{\varphi}_2 + \bar{z}\varphi_2' + z\bar{\varphi}_2' + \bar{\psi}_2 + \psi_2)]|_{n=0} + \frac{2\mu}{a}\lambda^2\left\{\sum_{k=1}^{\infty}\delta_k\alpha_k(\zeta)a_{k2} + \nu\sum_{p=1}^{\infty}\cos\rho_p\zeta\left[\frac{a}{R}(1+\rho_p)f_{p2} - \rho_p^2 f_{p3}\right]\right\} + \dots \quad (4.7)$$

$$\tau_{nz} = \frac{2\mu}{a}\nu\lambda\sum_{k=1}^{\infty}\gamma_k(\zeta)\delta_k a_{k2} + \frac{2\mu}{a}\nu\lambda^2\left\{\sum_{k=1}^{\infty}\gamma_k(\zeta)\left[\delta_k a_{k3} - \frac{a}{2R}a_{k2}\right] - \sum_{p=1}^{\infty}\rho_p\sin\rho_p\zeta f_{p2}'\right\} + \dots \quad (4.8)$$

Near the edge additional terms appear in Eqs. (4.6) and (4.7) which are of the same order in λ as the solution of the two-dimensional problem. In the stress τ_{nz} for small λ the additional terms play the basic role (cf. [3]). Therefore, as in plate bending, the estimation of the state of stress on the boundary from the engineering theory must be approached carefully. The problem of stress concentration should be examined separately.

A numerical investigation of the state of stress in plates has been performed in a number of cases on the basis of the theory presented here. Although numerical data were provided for 50 boundary layers, the accuracy necessary for practical purposes was attained when 10 boundary layers were used.

BIBLIOGRAPHY

1. Lur'e, A. I., Prostranstvennye zadachi teorii uprugosti (Three-Dimensional Problems of the Theory of Elasticity), Gostekhizdat, 1955.
2. Lur'e, A. I., K teorii tolstykh plit (The theory of thick plates), PMM Vol. 6, Nos. 2 and 3,, 1942.
3. Aksentian, O. K. and Vorovich, I. I., Napriazhennoe sostoianie plity maloi tolshchiny (The state of stress in thin plate), PMM Vol. 27, No. 6, 1963.
4. Leibenzon, L. S., Kurs teorii uprugosti (A Course in the Theory of Elasticity), Gostekhizdat, 1947.

Translated by A. R. R.